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Inferior mappings and applications

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Abstract. Using partially ordered sets, the inferior mapping is introduced, and its application to BCK/BCI-algebras is discussed. The notions of inferior subalgebras and (commutative) inferior ideals are introduced, and their relations and related properties are investigated. Conditions for an inferior ideal to be commutative are provided. An extension property for a commutative inferior ideal is established.

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1. Introduction

Algebras have played an important role in pure and applied mathematics and have its comprehensive applications in many aspects including dynamical systems and genetic code of biology (See [1, 2, 3, 4]). Starting from the four DNA bases order in the Boolean lattice, Sáanchez et al. [5] proposed a novel Lie Algebra of the genetic code which shows strong connections among algebraic relationship, codon assignments and physicochemical properties of amino acids. A BCK/BCI-algebra (See [6, 7, 8]) is an important class of logical algebras introduced by Iséki and was extensively investigated by several researchers. Jun and Song [9] introduced the notion of BCK-valued functions and investigated several properties. They established block-codes by using the notion of BCK-valued functions, and shown that every finite BCK-algebra determines a block-code.

The aim of this paper is to introduce the notion of inferior mapping by using partially ordered sets, and apply it to BCK/BCI-algebras. Using the inferior mapping, we introduce the notions of inferior subalgebras and (commutative) inferior ideals in BCK/BCI-algebras, and investigate related properties. We discuss relations among an inferior subalgebra, an inferior ideal and a commutative inferior ideal. We provide conditions for an inferior mapping to be an inferior ideal. We also provide conditions for an inferior ideal to be a commutative inferior ideal. We establish an extension property for a commutative inferior ideal.

2. Preliminaries

We display basic definitions and properties of BCK/BCI-algebras that will be used in this paper. For more details of BCK/BCI-algebras, we refer the reader to [6] and [8].

An algebra $\mathcal{X} := (X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra*, if it satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,
(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,
(III) $(\forall x \in X)\ (x * x = 0)$,
(IV) $(\forall x, y \in X)\ (x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra $\mathcal{X}$ satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then $\mathcal{X}$ is called a *BCK-algebra*.

Any BCK/BCI-algebra $\mathcal{X}$ satisfies the following conditions:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{2.1}$$

$$(\forall x, y, z \in X)\,(x \leq y \ \Rightarrow\ x * z \leq y * z,\ z * y \leq z * x)\,, \tag{2.2}$$

$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{2.3}$$

$$(\forall x, y, z \in X)\,((x * z) * (y * z) \leq x * y) \tag{2.4}$$

where $x \leq y$ if and only if $x * y = 0$.

A BCK-algebra $\mathcal{X}$ is said to be *commutative*, if $x \wedge y = y \wedge x$ for all $x, y \in X$, where $x \wedge y = y * (y * x)$.

A nonempty subset S of a BCK/BCI-algebra $\mathcal{X}$ is called a *subalgebra* of $\mathcal{X}$, if $x * y \in S$ for all $x, y \in S$. A subset A of a BCK/BCI-algebra $\mathcal{X}$ is called an *ideal* of $\mathcal{X}$, if it satisfies:

$$0 \in A, \tag{2.5}$$

$$(\forall x, y \in X)\,(x * y \in A,\ y \in A \ \Rightarrow\ x \in A)\,. \tag{2.6}$$

A subset A of a BCK-algebra $\mathcal{X}$ is called a *commutative ideal* of $\mathcal{X}$, if it satisfies (2.5) and

$$(\forall x, y, z \in X)\,((x * y) * z \in A,\ z \in A \ \Rightarrow\ x * (y * (y * x)) \in A)\,. \tag{2.7}$$

3. Inferior mappings

Let X be a nonempty set and let U be a partially ordered set with the partial ordering $\precsim$ and the last element θ. Then the statement

$a \precsim b$ is read "*a precedes b*"

In this context, we also write:

$b \succsim a$ means $a \precsim b$; and read "*b succeeds a*",
$a \prec b$ means $a \precsim b$ and $a \neq b$; and read "*a strictly precedes b*",
$b \succ a$ means $a \prec b$; and read "*b strictly succeeds a*".

We consider a pair (f, X) on $(U, \precsim)$, where $f : X \to \mathcal{P}(U)$ is a mapping and $\mathcal{P}(U)$ is the power set of U. Define a mapping

$$\tilde{f} : X \to U,\ x \mapsto \begin{cases} \inf f(x) & \text{if } \exists \inf f(x) \\ \theta & \text{if } \nexists \inf f(x) \text{ or } f(x) = \varnothing \end{cases} \tag{3.1}$$

which is called the *inferior mapping of* X *related to the pair* (f, X) on $(U, \precsim)$.

Example 3.1. Let $U = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$ be ordered by

$$x \precsim y \ \Leftrightarrow\ y \text{ divides } x.$$

The Hasse diagram of U appears in Figure 1.

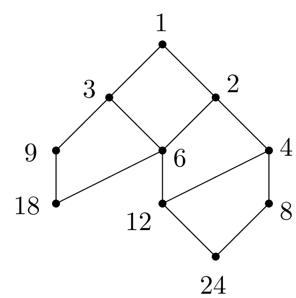

Figure 1

For a set $X = \{a, b, c, d\}$, let (f, X) be a pair on $(U, \precsim)$ where f is given as follows:

$$f : X \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{2, 3, 4, 6\} & \text{if } x = a \\ \{4, 6, 8, 12\} & \text{if } x = b \\ \{1, 3, 6, 9\} & \text{if } x = c \\ \{8, 12, 18\} & \text{if } x = d. \end{cases}$$

Then the inferior mapping $\tilde{f}$ of X related to the pair (f, X) on $(U, \precsim)$ is described as follows: $\tilde{f}(a) = 12$, $\tilde{f}(b) = 24$ and $\tilde{f}(c) = 18$, but $\tilde{f}(d) = 1$ because there does not exist the infimum of $f(d)$.

Example 3.2. For any positive integer m, we will let $\mathbf{D}_m$ denote the set of divisors of m ordered by divisibility. The Hasse diagram of

$$\mathbf{D}_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$$

appears in Figure 2.

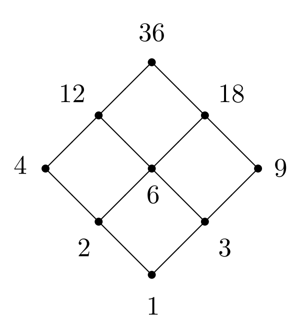

Figure 2

For a set $X = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, let (f, X) be a pair on $(U, \precsim)$ with $U = \mathbf{D}_{36}$ in which f is defined as follows:

$$f : X \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{1,2,3\} & \text{if } x = a_1 \\ \{2,3,6\} & \text{if } x = a_2 \\ \{6,9,12,18\} & \text{if } x = a_3 \\ \{12,36\} & \text{if } x = a_4 \\ \{4,6,9\} & \text{if } x = a_5 \\ \{2,6,12\} & \text{if } x = a_6. \end{cases}$$

Then the inferior mapping of X related to the pair (f, X) on $(U, \precsim)$ is described as follows: $\tilde{f}(a_1) = \tilde{f}(a_2) = \tilde{f}(a_5) = 1$, $\tilde{f}(a_3) = 3$, $\tilde{f}(a_4) = 12$ and $\tilde{f}(a_6) = 2$.

Example 3.3. Let $U = \{a, b, c, d, e, f\}$ be a set with the partial order "$\precsim$" as pictured in Figure 3.

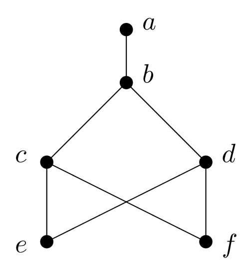

Figure 3

For a set $X = \{0, 1, 2, 3\}$, let (f, X) be a pair on $(U, \precsim)$ in which f is defined as follows:

$$f : X \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{a,b\} & \text{if } x = 0 \\ \{a,b,c\} & \text{if } x = 1 \\ \{b,c,d,e\} & \text{if } x = 2 \\ \{b,c,d,f\} & \text{if } x = 3. \end{cases}$$

Then the inferior mapping of X related to the pair (f, X) on $(U, \precsim)$ is described as follows: $\tilde{f}(0) = b$, $\tilde{f}(1) = c$, $\tilde{f}(2) = e$ and $\tilde{f}(3) = f$.

Example 3.4. Let $U = \{1, 2, 3, \cdots, 8\}$ be ordered as pictured in Figure 4.

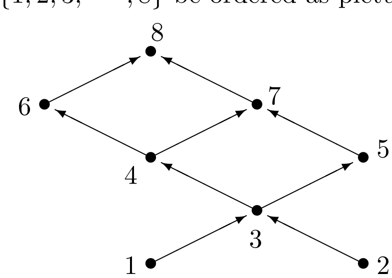

Figure 4

For a set $X = \{0, a, b, c, d\}$, let (f, X) be a pair on $(U, \precsim)$ in which f is defined as follows:

$$f : X \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{6,8\} & \text{if } x = 0 \\ \{4,6,7\} & \text{if } x = a \\ \{2,3,5,6,7\} & \text{if } x = b \\ \{3,4,5,6\} & \text{if } x = c \\ \{3,4,5,7\} & \text{if } x = d. \end{cases}$$

Then the inferior mapping of X related to the pair (f, X) on $(U, \precsim)$ is described as follows: $\tilde{f}(0) = 6$, $\tilde{f}(a) = 4$, $\tilde{f}(b) = 2$, and $\tilde{f}(c) = \tilde{f}(d) = 3$.

Let $\tilde{f}$ and $\tilde{g}$ be inferior mappings of X related to pairs (f, X) and (g, X), respectively, on $(U, \precsim)$. Then the union of (f, X) and (g, X) is defined to be the pair $(f \cup g, X)$ on $(U, \precsim)$ which is given as follows:

$$f \cup g : X \to \mathcal{P}(U),\ x \mapsto f(x) \cup g(x).$$

The intersection of (f, X) and (g, X) is defined to be the pair $(f \cap g, X)$ on $(U, \precsim)$ which is given as follows:

$$f \cap g : X \to \mathcal{P}(U),\ x \mapsto f(x) \cap g(x).$$

The inferior mapping of X related to the pair $(f \cup g, X)$ (resp. $(f \cap g, X)$) on $(U, \precsim)$ is called the union (resp. intersection) of $\tilde{f}$ and $\tilde{g}$ and is denoted by $\widetilde{f \cup g}$ (resp. $\widetilde{f \cap g}$). The inferior union of $\tilde{f}$ and $\tilde{g}$ is denoted by $\tilde{f} \Cup \tilde{g}$ and is defined by

$$\tilde{f} \Cup \tilde{g} : X \to U,\ x \mapsto \sup\{\tilde{f}(x), \tilde{g}(x)\}. \tag{3.2}$$

The inferior intersection of $\tilde{f}$ and $\tilde{g}$ is denoted by $\tilde{f} \Cap \tilde{g}$ and is defined as follows:

$$\tilde{f} \Cap \tilde{g} : X \to U,\ x \mapsto \inf\{\tilde{f}(x), \tilde{g}(x)\}, \tag{3.3}$$

where $(\tilde{f} \Cap \tilde{g})(x) = \theta$ if $\inf\{\tilde{f}(x), \tilde{g}(x)\}$ does not exist.

Example 3.5. Consider the poset $(U, \precsim)$ in Example 3.1. Let $\tilde{f}$ be an inferior mapping of $X = \{a, b, c, d\}$ which is given in Example 3.1, and let (g, X) be a pair on $(U, \precsim)$ where g is given as follows:

$$g : X \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{2,4,8\} & \text{if } x = a \\ \{1,3,6,9\} & \text{if } x = b \\ \{2,3,6\} & \text{if } x = c \\ \{3,6,4\} & \text{if } x = d. \end{cases}$$

Then the union and intersection of (f, X) and (g, X) are described as follows:

$$f \cup g : X \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{2,3,4,6,8\} & \text{if } x = a \\ \{1,3,4,6,8,9,12\} & \text{if } x = b \\ \{1,2,3,6,9\} & \text{if } x = c \\ \{3,6,4,8,12,18\} & \text{if } x = d. \end{cases}$$

and

$$f \cap g : X \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{2,4\} & \text{if } x = a \\ \{6\} & \text{if } x = b \\ \{3,6\} & \text{if } x = c \\ \varnothing & \text{if } x = d \end{cases}$$

respectively. The (infer) union and (infer) intersection of $\tilde{f}$ and $\tilde{g}$ are appeared in the Table 1.

TABLE 1. The (infer) union and (infer) intersection of $\tilde{f}$ and $\tilde{g}$

x	a	b	c	d
$\tilde{f}(x)$	12	24	18	1
$\tilde{g}(x)$	8	18	6	12
$(\tilde{f} \Cup \tilde{g})(x)$	4	6	6	1
$(\tilde{f} \Cap \tilde{g})(x)$	24	1	18	12
$\widetilde{f \cup g}(x)$	24	1	18	1
$\widetilde{f \cap g}(x)$	4	6	6	1

Let $\tilde{f}$ be an inferior mapping of X related to the pair (f, X) on $(U, \precsim)$. For any $\alpha \in U$, the sets

$S(\tilde{f}, \alpha) := \{x \in X \mid \tilde{f}(x) \text{ succeeds } \alpha\}$,

$P(\tilde{f}, \alpha) := \{x \in X \mid \tilde{f}(x) \text{ preceeds } \alpha\}$,

$SS(\tilde{f}, \alpha) := \{x \in X \mid \tilde{f}(x) \text{ strictly succeeds } \alpha\}$,

and

$SP(\tilde{f}, \alpha) := \{x \in X \mid \tilde{f}(x) \text{ strictly preceeds } \alpha\}$

are called the *upper α-inferior set,* the *lower α-inferior set,* the *strictly upper α-inferior set,* and the *strictly lower α-inferior set* of $\tilde{f}$, respectively.

Example 3.6. For the inferior mapping $\tilde{f}$ in Example 3.1, we have

(1) $$S(\tilde{f}, \alpha) = \begin{cases} \{d\} & \text{if } \alpha \in \{1, 2, 3, 4, 6, 8, 9\} \\ \{a, d\} & \text{if } \alpha = 12 \\ \{c, d\} & \text{if } \alpha = 18 \\ \{a, b, d\} & \text{if } \alpha = 24. \end{cases}$$

(2) $$P(\tilde{f}, \alpha) = \begin{cases} U & \text{if } \alpha = 1 \\ \{a, b, c\} & \text{if } \alpha \in \{2, 3\} \\ \{a, b\} & \text{if } \alpha \in \{4, 6, 12\} \\ \{c\} & \text{if } \alpha \in \{9, 18\} \\ \{b\} & \text{if } \alpha \in \{8, 24\}. \end{cases}$$

(3) $$SS(\tilde{f}, \alpha) = \begin{cases} \varnothing & \text{if } \alpha = 1 \\ \{d\} & \text{if } \alpha \in \{2, 3, 4, 6, 8, 9, 12, 18\} \\ \{a, d\} & \text{if } \alpha = 24. \end{cases}$$

(2) $$SP(\tilde{f}, \alpha) = \begin{cases} \{a, b, c\} & \text{if } \alpha \in \{1, 3\} \\ \{a, b\} & \text{if } \alpha \in \{2, 4, 6\} \\ \{b\} & \text{if } \alpha \in \{8, 12\} \\ \{c\} & \text{if } \alpha = 9, \\ \varnothing & \text{if } \alpha \in \{18, 24\}. \end{cases}$$

Obviously, we have

$$\begin{aligned}\tilde{f}(x) &= \inf\{\alpha \in U \mid x \in P(\tilde{f}, \alpha)\}\\ &= \sup\{\alpha \in U \mid x \in S(\tilde{f}, \alpha)\}.\end{aligned}$$

Proposition 3.7. *Let $\tilde{f}$ be an inferior mapping of X related to the pair (f, X) on $(U, \precsim)$. For any $\alpha \in U$, we have*

(1) $S(\tilde{f}, \varnothing) = P(\tilde{f}, U) = X$,
(2) $SS(\tilde{f}, \alpha) \subseteq S(\tilde{f}, \alpha)$ *and* $SP(\tilde{f}, \alpha) \subseteq P(\tilde{f}, \alpha)$,
(3) $SS(\tilde{f}, \alpha) = S(\tilde{f}, \alpha)$ *and* $SP(\tilde{f}, \alpha) = P(\tilde{f}, \alpha)$ *if and only if there is no* $x \in X$ *such that* $\tilde{f}(x) = \alpha$, *that is,* $\alpha \notin \mathrm{Im}(\tilde{f})$,
(4) *If* $\alpha, \beta \in \mathrm{Im}(\tilde{f})$ *and* $\alpha \neq \beta$, *then* $S(\tilde{f}, \alpha) \neq S(\tilde{f}, \beta)$ *and* $P(\tilde{f}, \alpha) \neq P(\tilde{f}, \beta)$,
(5) *For any* $\alpha, \beta \in \mathrm{Im}(\tilde{f})$, *if* α *precedes* β, *then* $SS(\tilde{f}, \alpha) \supseteq SS(\tilde{f}, \beta)$, $S(\tilde{f}, \alpha) \supseteq S(\tilde{f}, \beta)$, $P(\tilde{f}, \alpha) \subseteq P(\tilde{f}, \beta)$, *and* $SP(\tilde{f}, \alpha) \subseteq SP(\tilde{f}, \beta)$.
(6) $\tilde{f}(x) = \inf\{\alpha \in U \mid x \in P(\tilde{f}, \alpha)\} = \sup\{\alpha \in U \mid x \in S(\tilde{f}, \alpha)\}$ *for all* $x \in X$.

Proof. Straightforward. □

Proposition 3.8. *Let $\tilde{f}$ be an inferior mapping of X related to the pair (f, X) on $(U, \precsim)$. For any $\alpha, \beta \in U \setminus \mathrm{Im}(\tilde{f})$, if α strictly precedes β, then*

(1) $SS(\tilde{f}, \alpha) \supseteq SS(\tilde{f}, \beta)$ *and* $SP(\tilde{f}, \alpha) \subseteq SP(\tilde{f}, \beta)$,
(2) $S(\tilde{f}, \alpha) \supseteq S(\tilde{f}, \beta)$ *and* $P(\tilde{f}, \alpha) \subseteq P(\tilde{f}, \beta)$.

Proof. Assume that $\alpha, \beta \notin \mathrm{Im}(\tilde{f})$ and α strictly precedes β.

(1) If $x \in SS(\tilde{f}, \beta)$, then $\tilde{f}(x)$ strictly succeeds β. Thus $\tilde{f}(x)$ strictly succeeds α. So $x \in SS(\tilde{f}, \alpha)$. Hence $SS(\tilde{f}, \alpha) \supseteq SS(\tilde{f}, \beta)$. Now if $y \in SP(\tilde{f}, \alpha)$, then $\tilde{f}(y)$ strictly precedes α which implies that $\tilde{f}(y)$ strictly precedes β. Therefore $SP(\tilde{f}, \alpha) \subseteq SP(\tilde{f}, \beta)$.

(2) is induced by (1) and Proposition 3.7textcolorred3). □

4. Applications to BCK/BCI-algebras

Definition 4.1. Let $\mathcal{X} := (X, *, 0)$ be a BCK/BCI-algebra and let (f, X) be a pair on $(U, \precsim)$. By an *inferior subalgebra* of $\mathcal{X}$, we mean the inferior mapping $\tilde{f}$ of $\mathcal{X}$ related to the pair (f, X) on $(U, \precsim)$ such that

$$(\forall x, y \in X)\left(\tilde{f}(x * y) \text{ succeeds the infimum of } \tilde{f}(x) \text{ and } \tilde{f}(y)\right). \tag{4.1}$$

Example 4.2. Let $X = \{0, a, b, c\}$ be a set with a binary operation '$*$' shown in Table 2.

Then $\mathcal{X} := (X, *, 0)$ is a BCK-algebra (see [8]). Consider the poset $(U, \precsim)$ which is given in Example 3.1.

(1) Let (f, X) be a pair on $(U, \precsim)$ where f is given by

$$f : X \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{1, 2\} & \text{if } x = 0 \\ \{4, 6, 8\} & \text{if } x = a \\ \{2, 3, 4, 6\} & \text{if } x = b \\ \{1, 2, 3, 6\} & \text{if } x = c. \end{cases}$$

Table 2. Cayley table for the binary operation '$*$'

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	b
c	c	c	c	0

Then the inferior mapping of $\mathcal{X}$ related to the pair (f, X) on $(U, \precsim)$ is described as follows: $\tilde{f}(0) = 2$, $\tilde{f}(a) = 24$, $\tilde{f}(b) = 12$ and $\tilde{f}(c) = 6$, and it is an inferior subalgebra of $\mathcal{X}$.

(2) Let (g, X) be a pair on $(U, \precsim)$ in which g is provided as follows:

$$g : X \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{2,4,6\} & \text{if } x \in \{0, a\} \\ \{1,3,6,9\} & \text{if } x = b \\ \{4,6,8,12\} & \text{if } x = c. \end{cases}$$

Then the inferior mapping of $\mathcal{X}$ related to the pair (g, X) on $(U, \precsim)$ is described as follows: $\tilde{g}(0) = \tilde{g}(a) = 12$, $\tilde{g}(b) = 18$ and $\tilde{g}(c) = 24$, and it is not an inferior subalgebra of $\mathcal{X}$ since $\tilde{g}(b*b) = \tilde{g}(0) = 12$ and $\inf\{\tilde{g}(b), \tilde{g}(b)\} = 18$ are noncomparable and so $\tilde{g}(b * b)$ does not succeed the infimum of $\tilde{g}(b)$ and $\tilde{g}(b)$.

(3) Let (h, X) be a pair on $(U, \precsim)$ in which h is given as follows:

$$h : X \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{2,4,6\} & \text{if } x = 0 \\ \{8,12,18\} & \text{if } x = a \\ \{1,3,6,9\} & \text{if } x = b \\ \{2,3,9\} & \text{if } x = c. \end{cases}$$

Then the inferior mapping of $\mathcal{X}$ related to the pair (h, X) on $(U, \precsim)$ is described as follows: $\tilde{h}(0) = 12$, $\tilde{h}(a) = 1$, and $\tilde{h}(b) = \tilde{h}(c) = 18$. Since

$$\tilde{h}(a * a) = \tilde{h}(0) = 12 \not\succsim 1 = \inf\{\tilde{h}(a), \tilde{h}(a)\},$$

$\tilde{f}$ is not an inferior subalgebra of $\mathcal{X}$.

Example 4.3. Let $X = \{0, 1, 2, a, b\}$ be a set with a binary operation '$*$' shown in Table 3.

Table 3. Cayley table for the binary operation '$*$'

$*$	0	1	2	a	b
0	0	0	0	a	a
1	1	0	1	b	a
2	2	2	0	a	a
a	a	a	a	0	0
b	b	a	b	1	0

Then $\mathcal{X} := (X, *, 0)$ is a BCI-algebra (see [8]). Consider the poset $(U, \precsim)$ which is given in Example 3.2. Let (f, X) be a pair on $(U, \precsim)$ where f is defined by

$$f : X \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{36\} & \text{if } x = 0 \\ \{2,3,6\} & \text{if } x \in \{1, b\} \\ \{12,18\} & \text{if } x = 2 \\ \{3,6,9\} & \text{if } x = a. \end{cases}$$

Then the inferior mapping of $\mathcal{X}$ related to the pair (f, X) on $(U, \precsim)$ is described as follows: $\tilde{f}(0) = 36$, $\tilde{f}(a) = 3$, $\tilde{f}(b) = \tilde{f}(1) = 1$ and $\tilde{f}(2) = 6$, and it is an inferior subalgebra of $\mathcal{X}$.

Proposition 4.4. *If $\tilde{f}$ is an inferior subalgebra of a BCK/BCI-algebra $\mathcal{X}$, then $\tilde{f}(0)$ succeeds $\tilde{f}(x)$ for all $x \in X$.*

Proof. Since $x * x = 0$ for all $x \in X$, it is clear. □

Theorem 4.5. *If $\tilde{f}$ is an inferior subalgebra of a BCK/BCI-algebra $\mathcal{X}$, then the upper α-inferior set $S(\tilde{f}, \alpha)$ of $\tilde{f}$ is a subalgebra of $\mathcal{X}$ for all $\alpha \in U$ with $S(\tilde{f}, \alpha) \neq \varnothing$.*

Proof. Assume that $\tilde{f}$ is an inferior subalgebra of $\mathcal{X}$. Let $x, y \in S(\tilde{f}, \alpha)$. Then $\tilde{f}(x)$ and $\tilde{f}(y)$ succeed α. It follows from (4.1) that $\tilde{f}(x * y)$ succeeds α and that $x * y \in S(\tilde{f}, \alpha)$. Thus $S(\tilde{f}, \alpha)$ is a subalgebra of $\mathcal{X}$. □

Corollary 4.6. *If $\tilde{f}$ is an inferior subalgebra of a BCK/BCI-algebra $\mathcal{X}$, then the set*

$$A := \{x \in X \mid \tilde{f}(x) = \tilde{f}(0)\}$$

is a subalgebra of $\mathcal{X}$.

The following example illustrates Theorem 4.5.

Example 4.7. Consider the inferior subalgebra $\tilde{f}$ of $\mathcal{X}$ in Example 4.3. Then the upper α-inferior set $S(\tilde{f}, \alpha)$ of $\tilde{f}$ is a subalgebra of $\mathcal{X}$ for all $\alpha \in U$.

Theorem 4.8. *Let $\tilde{f}$ be an inferior mapping of a BCK/BCI-algebra $\mathcal{X}$ related to the pair (f, X) on $(U, \precsim)$ such that there exists the infimum of $\tilde{f}(x)$ and $\tilde{f}(y)$ for all $x, y \in X$. If the upper α-inferior set $S(\tilde{f}, \alpha)$ of $\tilde{f}$ is a subalgebra of $\mathcal{X}$ for all $\alpha \in U$ with $S(\tilde{f}, \alpha) \neq \varnothing$, then $\tilde{f}$ is an inferior subalgebra of $\mathcal{X}$.*

Proof. Let $x, y \in X$ and $\beta \in U$ be such that $\inf\{\tilde{f}(x), \tilde{f}(y)\} = \beta$. Then $\tilde{f}(x)$ and $\tilde{f}(y)$ succeed β, that is, $x, y \in S(\tilde{f}, \beta)$. Thus $x * y \in S(\tilde{f}, \beta)$. So $\tilde{f}(x * y) \succsim \beta = \inf\{\tilde{f}(x), \tilde{f}(y)\}$. Hence $\tilde{f}(x * y)$ succeeds the infimum of $\tilde{f}(x)$ and $\tilde{f}(y)$. Therefore $\tilde{f}$ is an inferior subalgebra of $\mathcal{X}$. □

Theorem 4.9. *Let $\tilde{f}$ and $\tilde{g}$ be inferior mappings of a BCK/BCI-algebra $\mathcal{X}$ related to the pair (f, X) and (g, X), respectively, on $(U, \precsim)$ such that there exists the infimum of $\tilde{f}(x)$ and $\tilde{g}(x)$ for all $x \in X$. If $\tilde{f}$ and $\tilde{g}$ are inferior subalgebras of $\mathcal{X}$, then the inferior intersection $\tilde{f} \Cap \tilde{g}$ of $\tilde{f}$ and $\tilde{g}$ is an inferior subalgebra of $\mathcal{X}$.*

Proof. Let $x, y \in X$. Then

$$\begin{aligned}(\tilde{f} \Cap \tilde{g})(x * y) &= \inf\{\tilde{f}(x * y), \tilde{g}(x * y)\} \\ &\succsim \inf\{\inf\{\tilde{f}(x), \tilde{f}(y)\}, \inf\{\tilde{g}(x), \tilde{g}(y)\}\} \\ &= \inf\{\inf\{\tilde{f}(x), \tilde{g}(x)\}, \inf\{\tilde{f}(y), \tilde{g}(y)\}\} \\ &= \inf\{(\tilde{f} \Cap \tilde{g})(x), (\tilde{f} \Cap \tilde{g})(y)\},\end{aligned}$$

that is, $(\tilde{f} \Cap \tilde{g})(x * y)$ succeeds the infimum of $(\tilde{f} \Cap \tilde{g})(x)$ and $(\tilde{f} \Cap \tilde{g})(y)$. Thus $\tilde{f} \Cap \tilde{g}$ is an inferior subalgebra of $\mathcal{X}$. □

Definition 4.10. Let $\mathcal{X} := (X, *, 0)$ be a BCK/BCI-algebra and let (f, X) be a pair on $(U, \precsim)$. By an *inferior ideal* of $\mathcal{X}$, we mean the inferior mapping $\tilde{f}$ of $\mathcal{X}$ related to the pair (f, X) on $(U, \precsim)$ which satisfies the following conditions:

(4.2) $\quad (\forall x \in X)\left(\tilde{f}(0) \text{ succeeds } \tilde{f}(x)\right),$

(4.3) $\quad (\forall x, y \in X)\left(\tilde{f}(x) \text{ succeeds the infimum of } \tilde{f}(x * y) \text{ and } \tilde{f}(y)\right).$

Example 4.11. Let $X = \{0, 1, 2, 3, 4\}$ be a set with a binary operation '$*$' shown in Table 4.

TABLE 4. Cayley table for the binary operation '$*$'

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

Then $\mathcal{X} := (X, *, 0)$ is a BCK-algebra (see [8]). Consider the poset $(U, \precsim)$ in Example 3.4. Let (f, X) be a pair on $(U, \precsim)$ where f is defined by

$$f : X \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{6, 8\} & \text{if } x = 0 \\ \{4, 6, 7\} & \text{if } x = 1 \\ \{4, 5, 7, 8\} & \text{if } x \in \{2, 3, 4\}. \end{cases}$$

Then the inferior mapping $\tilde{f}$ of $\mathcal{X}$ related to the pair (f, X) on $(U, \precsim)$ is described as follows: $\tilde{f}(0) = 6$, $\tilde{f}(1) = 4$ and $\tilde{f}(2) = \tilde{f}(3) = \tilde{f}(4) = 3$. It is routine to verify that $\tilde{f}$ is an inferior ideal of $\mathcal{X}$.

Example 4.12. Let $X = \{0, 1, 2, a\}$ be a set with a binary operation '$*$' shown in Table 5.

Table 5. Cayley table for the binary operation '$*$'

$*$	0	1	2	a
0	0	0	0	a
1	1	0	0	a
2	2	2	0	a
a	a	a	a	0

Then $\mathcal{X} := (X, *, 0)$ is a BCI-algebra (see [6]). Consider the poset $(U, \precsim)$ which is given in Example 3.1. Let (f, X) be a pair on $(U, \precsim)$ where f is defined by

$$f : X \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{1\} & \text{if } x = 0 \\ \{1, 2, 3\} & \text{if } x = 1 \\ \{2, 3, 4, 6\} & \text{if } x = 2 \\ \{4, 6, 8\} & \text{if } x = a. \end{cases}$$

Then the inferior mapping $\tilde{f}$ of $\mathcal{X}$ related to the pair (f, X) on $(U, \precsim)$ is described as follows: $\tilde{f}(0) = 1$, $\tilde{f}(1) = 6$ and $\tilde{f}(2) = 12$ and $\tilde{f}(3) = 24$. It is routine to verify that $\tilde{f}$ is an inferior ideal of $\mathcal{X}$.

Theorem 4.13. *If $\tilde{f}$ is an inferior ideal of a BCK/BCI-algebra $\mathcal{X}$, then the upper α-inferior set $S(\tilde{f}, \alpha)$ of $\tilde{f}$ is an ideal of $\mathcal{X}$ for all $\alpha \in U$ with $S(\tilde{f}, \alpha) \neq \varnothing$.*

Proof. Suppose that $\tilde{f}$ is an inferior ideal of $\mathcal{X}$. Let $\alpha \in U$ be such that $S(\tilde{f}, \alpha) \neq \varnothing$. Then there exists $x \in X$ such that $\tilde{f}(x)$ succeeds α. The transitivity of $\precsim$ and the condition (4.2) induces that $\tilde{f}(0)$ succeeds α. Thus $0 \in S(\tilde{f}, \alpha)$. Let $x, y \in X$ be such that $x * y \in S(\tilde{f}, \alpha)$ and $y \in S(\tilde{f}, \alpha)$. Then $\tilde{f}(x * y)$ and $\tilde{f}(y)$ succeed α. Thus the infimum of $\tilde{f}(x*y)$ and $\tilde{f}(y)$ succeed α. It follows from (4.3) and the transitivity of $\precsim$ that $\tilde{f}(x)$ succeeds α and that $x \in S(\tilde{f}, \alpha)$. So $S(\tilde{f}, \alpha)$ is an ideal of $\mathcal{X}$. □

Corollary 4.14. *If $\tilde{f}$ is an inferior ideal of a BCK/BCI-algebra $\mathcal{X}$, then the set*

$$A := \{x \in X \mid \tilde{f}(x) = \tilde{f}(0)\}$$

is an ideal of $\mathcal{X}$.

Theorem 4.15. *Let $\tilde{f}$ be an inferior mapping of a BCK-algebra $\mathcal{X}$ related to the pair (f, X) on $(U, \precsim)$ such that there exists the infimum of $\tilde{f}(x)$ and $\tilde{f}(y)$ for all $x, y \in X$. If the upper α-inferior set $S(\tilde{f}, \alpha)$ of $\tilde{f}$ is an ideal of $\mathcal{X}$ for all $\alpha \in U$ with $S(\tilde{f}, \alpha) \neq \varnothing$, then $\tilde{f}$ is an inferior ideal of $\mathcal{X}$.*

Proof. Assume that $S(\tilde{f}, \alpha)$ is an ideal of $\mathcal{X}$ for all $\alpha \in U$ with $S(\tilde{f}, \alpha) \neq \varnothing$. Then $S(\tilde{f}, \alpha)$ is a subalgebra of $\mathcal{X}$. Let $x, y \in X$. If we take β as the infimum of $\tilde{f}(x)$ and $\tilde{f}(y)$, then $x, y \in S(\tilde{f}, \beta)$. It follows that $x*y \in S(\tilde{f}, \beta)$ and so that $\tilde{f}(x*y)$ succeeds the infimum of $\tilde{f}(x)$ and $\tilde{f}(y)$. Thus $\tilde{f}$ is an inferior subalgebra of $\mathcal{X}$, and so $\tilde{f}(0)$ succeeds $\tilde{f}(x)$ for all $x \in X$ by Proposition 4.4. Now we take ε as the infimum of $\tilde{f}(x * y)$ and $\tilde{f}(y)$. Then $x * y \in S(\tilde{f}, \varepsilon)$ and $y \in \tilde{f}_\varepsilon$. Since $S(\tilde{f}, \varepsilon)$ is an ideal of $\mathcal{X}$, it follows that $x \in S(\tilde{f}, \varepsilon)$ and so that $\tilde{f}(x)$ succeeds the infimum of $\tilde{f}(x * y)$ and $\tilde{f}(y)$. Hence $\tilde{f}$ is an inferior ideal of $\mathcal{X}$. □

Theorem 4.16. *If $\tilde{f}$ and $\tilde{g}$ are inferior ideals of a BCK/BCI-algebra $\mathcal{X}$, then the inferior intersection $\tilde{f} \Cap \tilde{g}$ of $\tilde{f}$ and $\tilde{g}$ is an inferior ideal of $\mathcal{X}$.*

Proof. For any $x \in X$, we have

$$(\tilde{f} \Cap \tilde{g})(0) = \inf\{\tilde{f}(0), \tilde{g}(0)\} \succsim \inf\{\tilde{f}(x), \tilde{g}(x)\} = (\tilde{f} \Cap \tilde{g})(x),$$

that is, $(\tilde{f} \Cap \tilde{g})(0)$ succeeds $(\tilde{f} \Cap \tilde{g})(x)$. Now, let $x, y \in X$. Then

$$\begin{aligned}(\tilde{f} \Cap \tilde{g})(x) &= \inf\{\tilde{f}(x), \tilde{g}(x)\}\\ &\succsim \inf\{\inf\{\tilde{f}(x*y), \tilde{f}(y)\}, \inf\{\tilde{g}(x*y), \tilde{g}(y)\}\}\\ &= \inf\{\inf\{\tilde{f}(x*y), \tilde{g}(x*y)\}, \inf\{\tilde{f}(y), \tilde{g}(y)\}\}\\ &= \inf\{(\tilde{f} \Cap \tilde{g})(x*y), (\tilde{f} \Cap \tilde{g})(y)\}.\end{aligned}$$

Thus $(\tilde{f} \Cap \tilde{g})(x)$ succeeds the infimum of $(\tilde{f} \Cap \tilde{g})(x*y)$ and $(\tilde{f} \Cap \tilde{g})(y)$. So $\tilde{f} \Cap \tilde{g}$ is an inferior ideal of $\mathcal{X}$. □

Proposition 4.17. *Every inferior ideal $\tilde{f}$ of $\mathcal{X}$, where $\mathcal{X} := (X, *, 0)$ is a BCK/BCI-algebra, satisfies:*

$$(\forall x, y \in X)\left(x \leq y \;\Rightarrow\; \tilde{f}(x) \text{ succeeds } \tilde{f}(y)\right). \tag{4.4}$$

Proof. Let $x, y \in X$ be such that $x \leq y$. Using (4.3) and (4.2), we have

$$\tilde{f}(x) \succsim \inf\{\tilde{f}(x*y), \tilde{f}(y)\} = \inf\{\tilde{f}(0), \tilde{f}(y)\} = \tilde{f}(y),$$

that is, $\tilde{f}(x)$ succeeds $\tilde{f}(y)$ for all $x, y \in X$ with $x \leq y$. □

Theorem 4.18. *In a BCK-algebra $\mathcal{X}$, every inferior ideal of $\mathcal{X}$ is an inferior subalgebra on $\mathcal{X}$.*

Proof. Let $\tilde{f}$ be an inferior ideal of $\mathcal{X}$. Since $x * y \leq x$ for all $x, y \in X$, it follows from Proposition 4.17 and (4.3) that

$$\tilde{f}(x*y) \succsim \tilde{f}(x) \succsim \inf\{\tilde{f}(x*y), \tilde{f}(y)\} \succsim \inf\{\tilde{f}(x), \tilde{f}(y)\},$$

that is, $\tilde{f}(x*y)$ succeeds the infimum of $\tilde{f}(x)$ and $\tilde{f}(y)$. Then $\tilde{f}$ is an inferior subalgebra of $\mathcal{X}$. □

The converse of Theorem 4.18 may not be true as seen in the following example.

Example 4.19. Let $X = \{0, 1, 2, 3\}$ be a set with a binary operation '$*$' shown in Table 6.

Table 6. Cayley table for the binary operation '$*$'

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Then $\mathcal{X} := (X, *, 0)$ is a BCK-algebra (see [8]). Let $U = \{a, b, c, d, e, f, g\}$ be ordered as pictured in Figure 3.

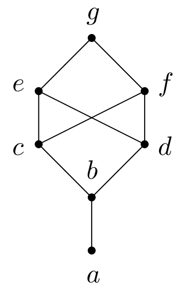

Figure 3

Let (f, X) be a pair on $(U, \precsim)$ where f is given as follows:

$$f : X \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{c, f\} & \text{if } x = 0 \\ \{b, d, f\} & \text{if } x = 2 \\ \{a, b, c, d\} & \text{if } x \in \{1, 3\}. \end{cases}$$

Then the inferior mapping $\tilde{f}$ of $\mathcal{X}$ related to the pair (f, X) on $(U, \precsim)$ is described as follows: $\tilde{f}(0) = c$, $\tilde{f}(2) = b$ and $\tilde{f}(1) = \tilde{f}(3) = a$. By routine calculations, we know that $\tilde{f}$ is an inferior subalgebra of $\mathcal{X}$, but it is not an inferior ideal of $\mathcal{X}$ because $\tilde{f}(1)$ does not succeed the infimum of $\tilde{f}(1 * 2)$ and $\tilde{f}(2)$.

Proposition 4.20. *Every inferior ideal $\tilde{f}$ of a BCK/BCI-algebra $\mathcal{X}$ satisfies the following assertion.*

$$(4.5)\quad (\forall x, y, z \in X) \Big(x * y \leq z \ \Rightarrow\ \tilde{f}(x) \text{ succeeds the infimum of } \tilde{f}(y) \text{ and } \tilde{f}(z) \Big).$$

Proof. Let $x, y, z \in X$ be such that $x * y \leq z$. Then $(x * y) * z = 0$, and so

$$\tilde{f}(x * y) \succsim \inf\{\tilde{f}((x * y) * z), \tilde{f}(z)\} = \inf\{\tilde{f}(0), \tilde{f}(z)\} = \tilde{f}(z)$$

by (4.3) and (4.2). It follows that

$$\tilde{f}(x) \succsim \inf\{\tilde{f}(x * y), \tilde{f}(y)\} \succsim \inf\{\tilde{f}(z), \tilde{f}(y)\}$$

which shows that $\tilde{f}(x)$ succeeds the infimum of $\tilde{f}(y)$ and $\tilde{f}(z)$ for all $x, y, z \in X$ with $x * y \leq z$. □

Theorem 4.21. *Let $\tilde{f}$ be the inferior mapping of a BCK/BCI-algebra $\mathcal{X}$ related to the pair (f, X) on $(U, \precsim)$. If $\tilde{f}$ satisfies two conditions (4.2) and (4.5), then $\tilde{f}$ is an inferior ideal of $\mathcal{X}$.*

Proof. Since $x * (x * y) \leq y$ for all $x, y \in X$, it follows from (4.5) that $\tilde{f}(x)$ succeeds the infimum of $\tilde{f}(x * y)$ and $\tilde{f}(y)$ for all $x, y \in X$. Then $\tilde{f}$ is an inferior ideal of $\mathcal{X}$. □

5. Commutative inferior ideals

Definition 5.1. Let $\mathcal{X} := (X, *, 0)$ be a BCK-algebra and let (f, X) be a pair on $(U, \precsim)$. By a *commutative inferior ideal* of $\mathcal{X}$, we mean the inferior mapping $\tilde{f}$ of $\mathcal{X}$ related to the pair (f, X) on $(U, \precsim)$ which satisfies the condition (4.2) and $\tilde{f}(x*(y*(y*x)))$ succeeds the infimum of $\tilde{f}((x*y)*z)$ and $\tilde{f}(z)$ for all $x, y, z \in X$.

Example 5.2. Let $U = \{1, 2, 3, \cdots, 9\}$ be ordered as pictured in Figure 4.

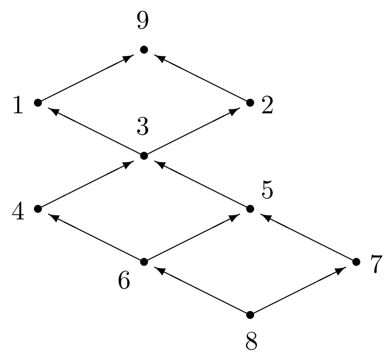

Figure 4

Let $X = \{0, a, b, c, d\}$ be a set with a binary operation '$*$' shown in Table 7.

Table 7. Cayley table for the binary operation '$*$'

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	b	0
c	c	a	c	0	c
d	d	d	d	d	0

Then $\mathcal{X} := (X, *, 0)$ is a BCK-algebra (see [8]). Let (f, X) be a pair on $(U, \precsim)$ where f is given as follows:

$$f : X \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{1, 2, 9\} & \text{if } x \in \{0, b\} \\ \{1, 3, 4, 5\} & \text{if } x = d \\ \{3, 5, 6, 7\} & \text{if } x \in \{a, c\}. \end{cases}$$

Then the inferior mapping $\tilde{f}$ of $\mathcal{X}$ related to the pair (f, X) on $(U, \precsim)$ is described as follows: $\tilde{f}(0) = \tilde{f}(b) = 3$, $\tilde{f}(d) = 6$, and $\tilde{f}(a) = \tilde{f}(c) = 8$. It is routine to check that $\tilde{f}$ is a commutative inferior ideal of $\mathcal{X}$.

Theorem 5.3. *If $\mathcal{X}$ is a BCK-algebra, then every commutative inferior ideal of $\mathcal{X}$ is an inferior ideal of $\mathcal{X}$.*

Proof. Let $\tilde{f}$ be a commutative inferior ideal of $\mathcal{X}$ where $\mathcal{X}$ is a BCK-algebra. Then

$$\begin{aligned}\tilde{f}(x) &= \tilde{f}(x * (0 * (0 * x))) \\ &\succsim \inf\{\tilde{f}((x * 0) * z), \tilde{f}(z)\} \\ &= \inf\{\tilde{f}(x * z), \tilde{f}(z)\},\end{aligned}$$

that is, $\tilde{f}(x)$ succeeds the infimum of $\tilde{f}(x * z)$ and $\tilde{f}(z)$ for all $x, z \in X$. Thus $\tilde{f}$ is a inferior ideal of $\mathcal{X}$. □

The following example shows that the converse of Theorem 5.3 is not true in general.

Example 5.4. Consider the BCK-algebra $\mathcal{X}$ and the poset $(U, \precsim)$ which are given in Examples 5.2 and 3.4, respectively. Let (f, X) be a pair on $(U, \precsim)$, where f is given as follows:

$$f : X \to \mathcal{P}(U),\ x \mapsto \begin{cases} \{7, 8\} & \text{if } x = 0 \\ \{5, 6, 7\} & \text{if } x = a \\ \{2, 3, 4, 5, 7\} & \text{if } x \in \{b, c, d\}. \end{cases}$$

Then the inferior mapping $\tilde{f}$ of $\mathcal{X}$ is described as follows: $\tilde{f}(0) = 7$, $\tilde{f}(a) = 3$ and $\tilde{f}(b) = \tilde{f}(c) = \tilde{f}(d) = 2$. Routine calculations show that $\tilde{f}$ is an inferior ideal of $\mathcal{X}$. But it is not a commutative inferior ideal of $\mathcal{X}$ since $\tilde{f}(b * (d * (d * b)))$ does not succeed the infimum of $\tilde{f}((b * d) * 0)$ and $\tilde{f}(0)$.

Proposition 5.5. *Let $\tilde{f}$ be a commutative inferior ideal of a BCK-algebra $\mathcal{X}$. Then $\tilde{f}(x * (y * (y * x)))$ succeeds $\tilde{f}(x * y)$ for all $x, y \in X$.*

Proof. Since $\tilde{f}(x * (y * (y * x)))$ succeeds the infimum of $\tilde{f}((x * y) * z)$ and $\tilde{f}(z)$ for all $x, y, z \in X$, taking $z = 0$ and using (4.2) and (2.1) induces the desired result. □

We provide conditions for an inferior ideal to be commutative.

Theorem 5.6. *Let $\tilde{f}$ be an inferior ideal of a BCK-algebra $\mathcal{X}$ such that $\tilde{f}(x * (y * (y * x)))$ succeeds $\tilde{f}(x * y)$ for all $x, y \in X$. Then $\tilde{f}$ is a commutative inferior ideal of $\mathcal{X}$.*

Proof. Assume that $\tilde{f}(x*(y*(y*x)))$ succeeds $\tilde{f}(x*y)$ for all $x, y \in X$. Then by (4.3), $\tilde{f}(x*y)$ succeeds the infimum of $\tilde{f}((x*y)*z)$ and $\tilde{f}(z)$. Thus $\tilde{f}(x*(y*(y*x)))$ succeeds the infimum of $\tilde{f}((x * y) * z)$ and $\tilde{f}(z)$ for all $x, y, z \in X$. So $\tilde{f}$ is a commutative inferior ideal of $\mathcal{X}$. □

Combining Theorems 4.21 and 5.6, we have the following corollary.

Corollary 5.7. *Let $\tilde{f}$ be the inferior mapping of a BCK-algebra $\mathcal{X}$ related to the pair (f, X) on $(U, \precsim)$. If $\tilde{f}$ satisfies (4.2), (4.5) and $\tilde{f}(x * (y * (y * x)))$ succeeds $\tilde{f}(x * y)$ for all $x, y \in X$, then $\tilde{f}$ is a commutative inferior ideal of $\mathcal{X}$.*

Theorem 5.8. *In a commutative BCK-algebra, every inferior ideal is a commutative inferior ideal.*

Proof. Let $\tilde{f}$ be an inferior ideal of $\mathcal{X}$, where $\mathcal{X}$ is a commutative BCK-algebra. Note that

$$\begin{aligned}&((x*(y*(y*x)))*((x*y)*z))*z\\&=((x*(y*(y*x)))*z)*((x*y)*z)\\&\leq(x*(y*(y*x)))*(x*y)\\&=(x*(x*y))*(y*(y*x))=0,\end{aligned}$$

that is, $(x*(y*(y*x)))*((x*y)*z)\leq z$ for all $x,y,z\in X$. It follows from Proposition 4.20 that $\tilde{f}(x*(y*(y*x)))$ succeeds the infimum of $\tilde{f}((x*y)*z)$ and $\tilde{f}(z)$ for all $x,y,z\in X$. Then $\tilde{f}$ is a commutative inferior ideal of $\mathcal{X}$. □

Corollary 5.9. *If a BCK-algebra $\mathcal{X}$ satisfies the following condition:*

$$(\forall x,y\in X)\,(x*(x*y)\leq y*(y*x))\,,\tag{5.1}$$

then every inferior ideal is a commutative inferior ideal.

Lemma 5.10 ([8])**.** *Let A be an ideal of a BCK-algebra $\mathcal{X}$. Then A is commutative if and only if the following assertion hods.*

$$(\forall x,y\in X)\,(x*y\in A\;\Rightarrow\;x*(y*(y*x))\in A)\,.\tag{5.2}$$

Theorem 5.11. *If $\tilde{f}$ is a commutative inferior ideal of a BCK-algebra $\mathcal{X}$, then the upper α-inferior set $S(\tilde{f},\alpha)$ of $\tilde{f}$ is a commutative ideal of $\mathcal{X}$ for all $\alpha\in U$ with $S(\tilde{f},\alpha)\neq\varnothing$.*

Proof. Assume that $\tilde{f}$ is a commutative inferior ideal of a BCK-algebra $\mathcal{X}$. Then $\tilde{f}$ is an inferior ideal of $\mathcal{X}$ by Theorem 5.3. Thus $S(\tilde{f},\alpha)$ is an ideal of $\mathcal{X}$ for all $\alpha\in U$. Let $x,y\in X$ be such that $x*y\in S(\tilde{f},\alpha)$. Then $\tilde{f}(x*y)$ succeeds α. Thus $\tilde{f}(x*(y*(y*x)))$ succeeds α by using Proposition 5.5, that is, $x*(y*(y*x))\in S(\tilde{f},\alpha)$. So by Lemma 5.10, $S(\tilde{f},\alpha)$ is a commutative ideal of $\mathcal{X}$ for all $\alpha\in U$ with $S(\tilde{f},\alpha)\neq\varnothing$. □

Theorem 5.12. *Let $\tilde{f}$ be an inferior mapping of a BCK-algebra $\mathcal{X}$ related to the pair (f,X) on $(U,\precsim)$ such that there exists the infimum of $\tilde{f}(x)$ and $\tilde{f}(y)$ for all $x,y\in X$. If then the upper α-inferior set $S(\tilde{f},\alpha)$ of $\tilde{f}$ is a commutative ideal of $\mathcal{X}$ for all $\alpha\in U$ with $S(\tilde{f},\alpha)\neq\varnothing$, then $\tilde{f}$ is a commutative inferior ideal of $\mathcal{X}$.*

Proof. Assume that $S(\tilde{f},\alpha)$ is a commutative ideal of $\mathcal{X}$ for all $\alpha\in U$ with $S(\tilde{f},\alpha)\neq\varnothing$. Then $S(\tilde{f},\alpha)$ is an ideal of $\mathcal{X}$. Thus $\tilde{f}$ is an inferior ideal of $\mathcal{X}$ by Theorem 4.15. For any $x,y\in X$, if we take $\tilde{f}(x*y)=\varepsilon$, then $x*y\in S(\tilde{f},\varepsilon)$. It follows from Lemma 5.10 that $x*(y*(y*x))\in S(\tilde{f},\varepsilon)$. Thus $\tilde{f}(x*(y*(y*x)))$ succeeds $\varepsilon=\tilde{f}(x*y)$. So $\tilde{f}$ is a commutative inferior ideal of $\mathcal{X}$ by Theorem 5.6. □

Theorem 5.13. (Extension property) *Let $\tilde{f}$ and $\tilde{g}$ be inferior ideals of a BCK-algebra $\mathcal{X}$ such that $\tilde{f}(0)=\tilde{g}(0)$ and $\tilde{f}(x)$ precedes $\tilde{g}(x)$ for all $x(\neq 0)\in X$. If $\tilde{f}$ is a commutative inferior ideal of $\mathcal{X}$, then so is $\tilde{g}$.*

Proof. For any $x, y \in X$, let $u = x * y$. Then

$$\begin{aligned}\tilde{g}((x*u)*(y*(y*(x*u)))) &\succsim \tilde{f}((x*u)*(y*(y*(x*u)))) \\ &\succsim \tilde{f}((x*u)*y) = \tilde{f}((x*y)*u) \\ &= \tilde{f}(0) = \tilde{g}(0),\end{aligned}$$

which implies from (4.2) and the antisymmetry of $\precsim$ that

$$\tilde{g}((x*u)*(y*(y*(x*u)))) = \tilde{g}(0).$$

Note that

$$\begin{aligned}&(x*(y*(y*x)))*(x*(y*(y*(x*u)))) \\ &\leq (y*(y*(x*u)))*(y*(y*x)) \\ &\leq (y*x)*(y*(x*u)) \\ &\leq (x*u)*x = 0*u = 0,\end{aligned}$$

and thus $(x*(y*(y*x)))*(x*(y*(y*(x*u)))) = 0$. It follows from (4.3), (4.2) and (2.3) that

$$\begin{aligned}\tilde{g}(x*(y*(y*x))) &\succsim \inf\{\tilde{g}((x*(y*(y*x)))*(x*(y*(y*(x*u))))), \\ &\qquad \tilde{g}(x*(y*(y*(x*u))))\} \\ &= \inf\{\tilde{g}(0), \tilde{g}(x*(y*(y*(x*u))))\} \\ &= \tilde{g}(x*(y*(y*(x*u)))) \\ &\succsim \inf\{\tilde{g}((x*(y*(y*(x*u))))*u), \tilde{g}(u)\} \\ &= \inf\{\tilde{g}((x*u)*(y*(y*(x*u)))), \tilde{g}(u)\} \\ &= \inf\{\tilde{g}(0), \tilde{g}(u)\} \\ &= \tilde{g}(u) = \tilde{g}(x*y),\end{aligned}$$

that is, $\tilde{g}(x*(y*(y*x)))$ succeeds $\tilde{g}(x*y)$. So $\tilde{g}$ is a commutative inferior ideal of $\mathcal{X}$ by Theorem 5.6. □

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